Tangent Space and its differential

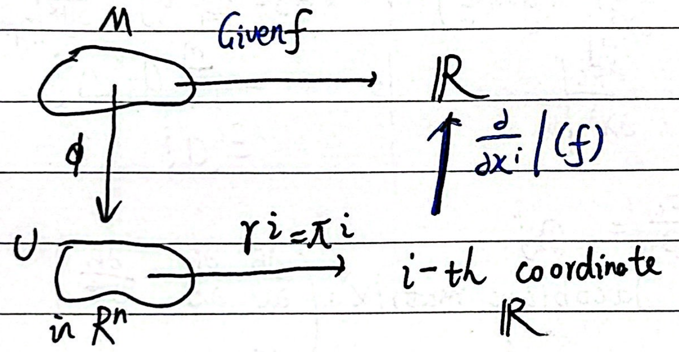
DEF(germ) A germ of a C^∞-function at p in M is an equivalence class of C^∞ functions defined in a neighborhood of p in M. (等价关系: if they agree on some small neighborhood of p). The set of ~~all~~ germs of C^∞-real valued functions at p is denoted by $C_p^\infty(M)$.

(DEF) Drivation / ~~differential~~: Tangent Space

A differential at a point is a linear map: $D: C_p^\infty(M) \to \mathbb{R}$, 满足 $D(fg)=(Df)g(p)+(Dg)f(p)$. 也被称为一个 ~~derivation~~ / tangent vector, 它们全体构成一个 algebra / vector space, 被称为切空间.

下面, 我们寻找 Tangent Space 之 basis

Given a coordinate neighborhood $(U,\phi)=(U,x^1,x^2,\dots,x^n)$ about a point p in a ~~differential~~ manifold M, $\frac{\partial}{\partial x^i}: C_p^\infty(M) \to \mathbb{R}$ 定义为:



(DEF): The differential of a Map.

如若 $F: N \to M$ 为两个流形之间的 C^∞-map, 那么 $F_*: T_pN \to T_{F(p)}M$ 被称为一个 differential, 也就是说, 对两个流形之间的~~线~~ C^∞-map 做了线性化.

$$N \xrightarrow{F} M \quad \xmapsto{\text{induce}} \quad T_pN \xrightarrow{F_*} T_pM$$

~~[illegible]~~: 其中 $F_*(X_p) = C^\infty_{F(p)} \to \mathbb{R}$, 使得 $F_*(X_p)(g) = X_p(g \circ F)$.

F_* 的性质:
① 它是 linear map from T_pN to T_pM
② Chain rule: $N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3$

$$\rightsquigarrow$$

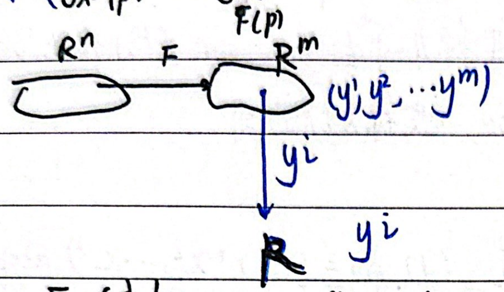
$$T_pN_1 \xrightarrow{f_{1*}} T_pN_2 \xrightarrow{f_{2*}} T_pN_3$$

既然它是一个linear map，而且，$T_pN \longrightarrow T_qM$ 对应的线性空间以 $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1,2\dots n}$ 与 $\left\{\frac{\partial}{\partial y^j}\right\}_{j=1,2\dots m}$ 为basis，我们自然而然地想知道这个线性映射在此两组"最自然"的基下的转换映射是什么！是什么呢？

- $F:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ smooth，$(x^1,x^2,\dots x^n)\longmapsto (y^1,y^2,\dots y^m)$

i.e. $F_*\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \sum_{k=1}^{m} a_j^k \frac{\partial}{\partial y^k}\Big|_{F(p)}$，求 a_j^k.

$F_*\left(\frac{\partial}{\partial x^j}\Big|_p\right): C^\infty_{F(p)} \longrightarrow \mathbb{R}$. 其中 $y^i \in C^\infty_{F(p)}$



- $F_*\left(\frac{\partial}{\partial x^j}\Big|_p\right)(y^i) = \sum_{k=1}^{m} a_j^k \frac{\partial}{\partial y^k}\Big|_{F(p)}(y^i)$.

$$LHS = F_*\left(\frac{\partial}{\partial x^j}\Big|_p\right)(y^i) = \frac{\partial}{\partial x^j}\Big|_p (y^i\circ F) = \frac{\partial F_i}{\partial x^j}\Big|_p.$$

$$RHS = \sum_{k=1}^{m} a_j^k \frac{\partial}{\partial y^k}\Big|_{F(p)}(y^i) = \sum_{k=1}^{m} a_j^k \Delta_k^i = a_j^i$$

Compare both sides: $\Rightarrow \frac{\partial F_i}{\partial x^j} = a_j^i$

回顾:Given $F:\mathbb{R}^n\longrightarrow\mathbb{R}^m$, Jacobian matrix:
$$\begin{pmatrix} \frac{\partial F_1}{\partial x^1} & \frac{\partial F_1}{\partial x^2} & \frac{\partial F_1}{\partial x^n} \\ \frac{\partial F_m}{\partial x^1} & \frac{\partial F_m}{\partial x^2} & \frac{\partial F_m}{\partial x^n} \end{pmatrix}$$

$$\updownarrow \qquad \|$$

$F_*: T_p\mathbb{R}^n \longrightarrow T_p\mathbb{R}^m$ 在给定基 $\left\{\left(\frac{\partial}{\partial x^i}\right)\Big|_p\right\}_{i=1}^n$；$\left\{\left(\frac{\partial}{\partial y^j}\right)\Big|_{F(p)}\right\}_{j=1}^m$ 下之转换阵

$$\begin{pmatrix} \frac{\partial F_1}{\partial x^1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_m}{\partial x^1} & \frac{\partial F_m}{\partial x^2} & \frac{\partial F_m}{\partial x^n} \end{pmatrix}$$

- Chain rule (Some functoriality...)

$$N \xrightarrow{F} M \xrightarrow{G} P$$

$$T_pN \xrightarrow{F_{*,p}} T_{F(p)}M \xrightarrow{G_{*,F(p)}} T_{G(F(p))}P$$

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Chair rule:

$N \xrightarrow{F} M \xrightarrow{G} P$ are smooth maps of manifolds, $p \in N$. then

$(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$.

proof:

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The tangent Space

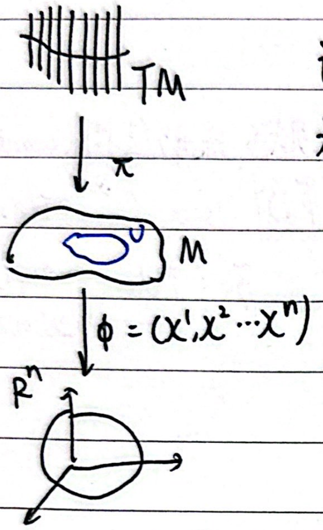
12.1 The topology of the Tangle bundle

$TM = \bigcup_{p\in M} T_pM = \coprod_{p\in M} T_pM$ (这是由于对$p\neq q$, 自然的有$T_pM \neq T_qM$)

$\pi: TM \longrightarrow M: v \longmapsto p$, if $v\in T_pM$, 这是一个natural map.

我们希望给TM赋予拓扑结构:

Given $(U,\phi) = (U, x^1, x^2, \cdots x^n)$ is a coordinate chart on M. 令 $TU = \bigcup_{p\in U} T_pU$
$= \bigcup_{p\in U} T_pM$ (由于U为M中的开集, 我们知道$T_pU = T_pM$)



记 $\bar{x}^i = x^i \circ \pi$

$\longrightarrow \mathbb{R}^{2n}$ 的子流形

构造 $\tilde{\phi}: TU \longrightarrow \phi(U)\times\mathbb{R}^n$. 其中$v\in T_pU = T_pM = \sum c^i \frac{\partial}{\partial x^i}\Big|_p$

$v \longmapsto (x^1(p), x^2(p), \cdots x^n(p), c^1(v), c^2(v), \cdots c^n(v))$

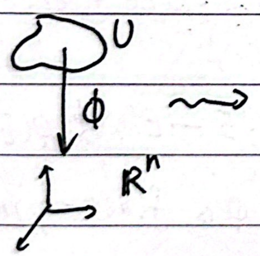
$\tilde{\phi}$ 是双射. 那么通过$\tilde{\phi}$将$\phi(U)\times\mathbb{R}^n$的子拓扑transfer到TU上

$\tilde{\phi}^{-1}: \phi(U)\times\mathbb{R}^n \longrightarrow TU$

$(\phi(p), c^1, c^2 \cdots c^n) \mapsto \sum c^i \frac{\partial}{\partial x^i}\Big|_p$

若 $V\subseteq U$为子空间. $\phi(V)\times\mathbb{R}^n$为$\phi(U)\times\mathbb{R}^n$的子空间.

$\Rightarrow$ the relative topology on TV as a subset of TU is same as the topology induced by $\tilde{\phi}|_{TV}: TV \longrightarrow \phi(V)\times\mathbb{R}^n$



ϕ_*

$T_pU \ni v$

$T_p\mathbb{R}^n \cong \mathbb{R}^n \quad \langle c^1, c^2 \cdots c^n\rangle = c^1\frac{\partial}{\partial x^1} + c^2\frac{\partial}{\partial x^2} + \cdots + c^n\frac{\partial}{\partial x^n}$

规定 $\mathcal{B} = \bigcup_\alpha \{A \mid A \text{ open in } T(U_\alpha), U_\alpha \text{ is a coordinate open set in } M\}$.

Lemma 12.1 ① ∀ M is a manifold. TM is the union of all $A\in\mathcal{B}$

证明: $TM = \bigcup_\alpha T(U_\alpha) \subseteq \bigcup_{A\in\mathcal{B}} A \subseteq TM$

② U, V为coordinate open sets. 若 A is open in TU; B open in TV, then $A\cap B$ is open in $T(U\cap V)$:

$\Rightarrow$ 因之. $\mathcal{B}$为TM的拓扑基

$\tilde{\phi}(A)$ is open subset of $\phi(U)\times\mathbb{R}^n$. $\tilde{\phi}(B)$ is open in $\phi(V)\times\mathbb{R}^n$

$\tilde{\phi}(A\cap B): (\phi(U)\cap\phi(V))\times\mathbb{R}^n$.

Lemma 12.2. A manifold M has a countable basis consisting of coordinate open sets

证: $\mathcal{B}=\{B_i\}$. M上极大之 atlas 为 $\{(U_\alpha,\phi_\alpha)\}$.

$\forall p\in U_\alpha$, 可找到一个 $B_{p,\alpha}\in\mathcal{B}$ 使 $p\in B_{p,\alpha}\subset U_\alpha$.

prop 12.3 The tangent bundle TM of a manifold M is second countable.

prop 12.4 The tangent bundle TM of a manifold M is Hausdorff.

12.2. The Manifold structure on Tangent bundle

在将 Tangent bundle T_pM 这个集合赋予拓扑后, 再给它赋予流形结构 (coordinate maps)

$\{(U_\alpha,\phi_\alpha)\}$ 为 a C^∞-atlas for M. then $\{(TU_\alpha,\tilde{\phi}_\alpha)\}$ is a C^∞ atlas for the tangent bundle TM. 其中 $\tilde{\phi}_\alpha$ is a map on TU_α. $\tilde{\phi}_\alpha: TU_\alpha \longrightarrow \phi(U_\alpha)\times\mathbb{R}^n$

$v \longmapsto (x^1(p), x^2(p), \cdots x^n(p), c^1$

满足 $\begin{cases} TM=\bigcup_\alpha TU_\alpha \\ (TU_\alpha)\cap(TU_\beta),\ \tilde{\phi}_\alpha \text{ and } \tilde{\phi}_\beta \text{ are } C^\infty\text{-compatible.}\end{cases}$

12.3. Vector bundles

On the tangent bundle TM of a smooth manifold M, the natural projection $\pi: TM\longrightarrow M$ $\pi(p,v)=p$ maps TM into a C^∞ vector bundle over M.

- fiber: Given any map $\pi: E\to M$, $\pi^{-1}(p)=\pi^{-1}(\{p\})$ 称为是p点的 fiber, 写为 E_p
- fiber-preserving map:

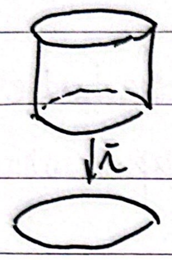
$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

$\forall$ two maps $E\xrightarrow{\pi}M$, $E'\xrightarrow{\pi'}M$, $\exists\ \phi: E\to E'$, 使 $\phi(E_p)\subseteq E'_p$, for any $p\in M$, then we say ϕ a fiber-preserving map.

$\pi: E\to M$ 为 surjective smooth map between manifolds, say ~~they are~~ it's locally trivial of rank r, if: ①. $\pi^{-1}(p)$ has the structure of a vector space of dim r.

②. $\forall p\in M$ $\exists$ 包含 p 之邻域 U, 以及 a fiber preserving 微分同胚 $\phi: \pi^{-1}(U)\longrightarrow U\times\mathbb{R}^r$. 使 $\forall q$, $\phi|_{\pi^{-1}(q)}: \pi^{-1}(q)\to\{q\}\times\mathbb{R}^r$ 为线性空间的同构

- A C^∞ vector bundle of rank r : (E, M, π), $E \xrightarrow{\pi} M$ and π is a localy trivial map of rank r. The ~~total~~ manifold E is called total space of the vector bundle, M is called the base space. E is a vector bundle over M.
- $S \subseteq M$为正则子流形. 那么 $(\pi^{-1}S, S, \pi|_{\pi^{-1}S})$ is a C^∞ vector bundle over S, called the restriction of E to S, as $E|_S$.



a circulor cylinder
is a product bundle over a circle

- The tangent bundle of a manifold M is the triple (TM, M, π). TM is the total space of the tangent bundle. TM is also referred to as the tangent bundle.
(我的猜法:在一点处粘了一个线性空间)

Examples.

$\pi: M \times \mathbb{R}^r \to M$ is a vector bundle of rank r. called the product bundle of rank r over M.

DEF: (bundle map) Let $\pi_E: E \to M$, $\pi_F: F \to N$ be 2 vector bundles, possibly of different ranks. A bundle map from E to F is a pair of maps $(f, \tilde{f})$:
其中 $M \xrightarrow{f} N$, $E \xrightarrow{\tilde{f}} F$, 成立:

①
$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

②. $\tilde{f}$ is linear on each fiber. 即: $\forall p \in M$; $E_p \xrightarrow{\tilde{f}} F_{f(p)}$ is a linear map of vector spaces.

比如说，$N \xrightarrow{f} M$ is a smooth map, 则它 induce a bundle map:

$$\tilde{f}(p,v) = \{f(p)\} \times T_{f(p)}M \subseteq TM, \ \forall v \in T_pN$$

$$\begin{tikzcd} (p,v) & TN \arrow[r, "\tilde{f}"] \arrow[d, "\pi_N"] & TM \arrow[d, "\pi_M"] \\ & N \arrow[r, "f"] & M \\ p \arrow[rr, mapsto] & & f(p) \end{tikzcd}$$

- 若 E, F are 2 vector bundles over the same manifold M, then ~~the~~ a bundle map from E to F over M is a bundle map in which the base map is the identity $\mathbb{1}_M$.

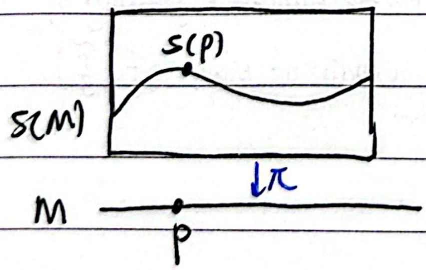
$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d] & F \arrow[d] \\ M \arrow[r, "\mathrm{Id}"] & M \end{tikzcd}$$

现固定M，考虑M上所有的C^∞ vector bundle, and C^∞ bundle maps over M. In this category it makes sense to speak of isomorphism of vector bundles over M. Any vector bundle over M isomorphic over M to the product bundle $M\times\mathbb{R}^r$ is called a trivial bundle.

Smooth sections

- A section of a vector bundle $\pi: E \to M$ is a map $s: M \to E$ s.t. $\pi\circ s = \mathbb{1}_M$, the identity function of M. Say a section is smooth if it's smooth as a map from M to E.

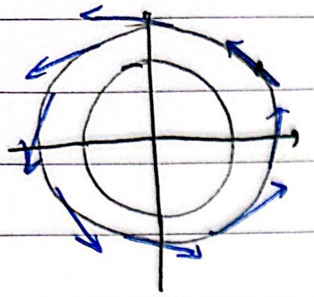
$$\begin{tikzcd} E \arrow[d, "\pi"] \\ M \end{tikzcd}$$



A vector field X on a manifold M is a function that assigns a tangent vector $X_p \in T_pM$ to each $p\in M$. 用tangent bundle 的说法来讲, a vector field on M is simply a section of ~~the~~ the tangent bundle $\pi: TM \to M$. The vector field is smooth, if the map is smooth from M to TM.

$$M \longrightarrow TM \longrightarrow M$$
$$p \longmapsto T_pM \longrightarrow p$$

例子: $X_{(x,y)} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \begin{bmatrix}-y\\x\end{bmatrix}$ defines a smooth vector field on $\mathbb{R}^2$.



$$\mathbb{R}^2 \longrightarrow X_{\mathbb{R}^2} = \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$p \longmapsto \begin{bmatrix}-y_p\\x_p\end{bmatrix} \longrightarrow p$$

prop: s, t are C^∞ sections of a C^∞ vector bundle $\pi: E \longrightarrow M$ and Let f be a C^∞ real valued function on M. 那么

① $s+t: M \longrightarrow E$ is def by: $(s+t)(p) = s(p) + t(p) \in E_p$

② $st: M \longrightarrow E$ is def by: $(st)(p) = s(p)t(p) \in E_p$.

Smooth Frames

$$\begin{matrix} E_p & & E \\ \uparrow & & \downarrow \pi \\ p & & M \end{matrix}$$

, A frame for vector bundle $E \xrightarrow{\pi} M$ over an open set U 为 collection of sections $s_1, s_2, \dots, s_r$ of E over U, s.t. $\forall p \in U$. the elements $s_1(p), s_2(p), \dots, s_r(p)$ forms a basis for the fiber $E_p = \pi^{-1}(p)$.

A frame for the tangent bundle $TM \longrightarrow M$ over an open set U is Simple called a frame on U.

Differential forms 参考: An intro to manifolds

Main ideas: Differential forms are intrinsic objects ass to manifold
Differential forms has a richer algebraic structure —微分分次代数

§17. Differential 1-forms

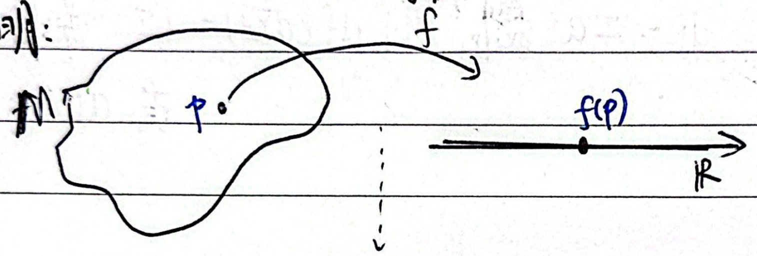
A differential 1-form 给流形 M 的每一个点上都赋予了一个余切向量 ω_p
余切向量场会在人们对于向量场的研究中自然出现: $\forall p\in\mathbb{R}^n$, $X_p=\sum a^i\frac{\partial}{\partial x^i}\Big|_p$, 把 a^i 单拎出来, $X_p\longrightarrow a^i$, actually 这将构成 $T_pM\to\mathbb{R}$ 的 ω_p.

定义: $f\in C^\infty(M)$, its differential is def as $(df)_p(X_p)=X_pf$, $\forall X_p\in T_pM$

性质: $f: M\to\mathbb{R}$ is a C^∞ function, then for $p\in M$ and $X_p\in T_pM$, 均有:

$$f_*(X_p)=(df)_p(X_p)\frac{d}{dt}\Big|_{f(p)}$$

证明:



$$T_pM \xrightarrow{f_*} T_{f(p)}\mathbb{R}\simeq\mathbb{R}$$

$f_*: T_pM\longrightarrow T_{f(p)}\mathbb{R}$, 以 $\frac{d}{dt}\Big|_{f(p)}$ 为基底

$X_p\longmapsto a\cdot\frac{\partial}{\partial t}\Big|_{f(p)}$

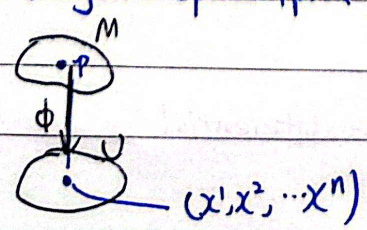
意识到 $f_*(X_p)(t)=a\cdot\frac{\partial t}{\partial t}=a=X_p(t\circ f)=X_p(f)=(df)_p(X_p)$

因之: $f_*(X_p)=(df)_p(X_p)\frac{d}{dt}\Big|_{f(p)}$

Local Expression for a Differential 1-form

在本节中令 $(U,\phi)=(U,x^1,x^2,\dots,x^n)$ 为 coordinate chart on a manifold M. Then $dx^1,dx^2\dots dx^n$ are 1-forms on U.

性质: At each point $p\in U$, the covectors $(dx^1)_p,\dots(dx^n)_p$ form a basis for the cotangent space T_p^*M dual to the basis $\partial/\partial x^1|_p,\dots,\partial/\partial x^n|_p$ for the tangent space T_pM.



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也就是说， T_pM 中的base $\left(\frac{\partial}{\partial x^1}\Big|_p\right), \left(\frac{\partial}{\partial x^2}\Big|_p\right), \cdots \left(\frac{\partial}{\partial x^n}\Big|_p\right)$

$\updownarrow \quad \updownarrow \quad \updownarrow$

differential 1-forms $(dx^1)_p \quad (dx^2)_p \quad (dx^n)_p$

满足 $(dx^i)_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \Delta_{ij}$

由此，我得到了 1-forms 之 bases $(dx^1)_p, (dx^2)_p, \cdots (dx^n)_p$

为啥呢？

$w: T_pM \to \mathbb{R}$

actually $\forall\, w \in$ 1-forms, w 完全由其在 $\left(\frac{\partial}{\partial x^1}\right), \left(\frac{\partial}{\partial x^2}\right), \cdots \left(\frac{\partial}{\partial x^n}\Big|_p\right)$ 上的作用来决定

而设 $w\left(\frac{\partial}{\partial x^i}\right) = a^i$. 则 w 与 $\sum a^i (dx^i)_p$ 在 T_pM 的base中的作用是相同的

$\Rightarrow w = \sum_{i=1}^{n} a^i dx^i$, 其中 a^i 为 functions on U.

若 differential 1-form $df = \sum a^i dx^i$ 则 $df\left(\frac{\partial}{\partial x^j}\right) = \left(\sum a^i dx^i\right)\left(\frac{\partial}{\partial x^j}\right)_p$

$= \sum_{i=1}^{n} a_i \Delta_{ij} = a_j \Rightarrow$

$\frac{\partial f}{\partial x^j} = a_j$.

Pullback of 1-Forms

$$\begin{array}{c} N \xrightarrow{F} M \\ \downarrow \\ T_pN \xrightarrow{F_{*p}} T_{F(p)}M \\ \downarrow \\ T_{F(p)}M^* \xrightarrow{\boxed{F_{*p}^{\vee}}} T_pN^* \end{array}$$

(push) forward

(pull back)

也写为 F^*

$F_{*p}(X_p) \in T_{F(p)}(M)$, $F_{*p}(X_p)(g) = X_p(g \circ F)$

$(F_{*p})^{\vee}(W_{F(p)}) \in T_pN^*$, $(F_{*p})^{\vee}(W_{F(p)})(X_p) = W_{F(p)}(F_{*p}(X_p))$

unlike vector fields, which is in general cannot be pushed forward under a C^∞ map, every covector field can be pulled back by a C^∞ map.

比如，给定一个 1-form on M, its pull back F^*w is a 1-form on N defined pointwisely by $(F^*w)_p = F^*(w_{F(p)})$

接下来我们研究 pullback of differential 的性质

(1). Commutations of the pullback with the differential

$N \xrightarrow{F} M$ is a C^∞-map. $\forall h \in C^\infty(M)$, $F^*(dh) = d(F^*h)$

LHS: $F^*(dh)(X_p) = dh(F_*(X_p))$ (def of pullback)

$= F_*(X_p)(h)$ (def of differential)

$= X_p(h \circ F)$ (def of F_*)

RHS: $(dF^*h)_p(X_p) = X_p(F^*h)$ (def of d as a function)

$= X_p(h \circ F)$ (def of F^* as a function)

(2). Pullback of a sum and a product